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NOTE ON THE DEFINITION OF AN ASYMPTOTE.

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In teaching the subject of asymptotes to a class of sophomore students recently, my attention was called to a point that seemed worth noting, concerning certain definitions of an asymptote to a curve. Text-books in common use differ considerably in their definitions of an asymptote. There are three different definitions that I find by an examination of text-books.

Judging from these books, it seems that a definition often given is the following:

(1) If the tangent to a curve approaches a limiting position, as the distance of the point of contact from the origin is indefinitely increased, this limiting position is called an asymptote. More briefly, it is sometimes stated that an asymptote to a curve is a tangent line whose point of contact is at infinity, but such that the line is not entirely at infinity.

A second definition in rather common use is the following:

(2) An asymptote to a curve is a straight line whose distance from a point on the curve diminishes indefinitely as the point moves along the curve to an infinite distance from the origin.

A third definition sometimes given is:

(3) If a straight line cuts a curve at two points at an infinite distance from the origin, but is not entirely at infinity, the line is called an asymptote.

The book that I am using in the class referred to above, gives the first of these three definitions. It is then shown how to determine the oblique asymptotes of the curve

$$f(x, y)=0, \quad f(x, y) \text{ being a polynomial of degree } n,$$

by selecting m and b in the straight line

$$y=mx+b,$$

so that at least two of the intersections of this line with $f(x, y)=0$ shall be at an infinite distance from the origin.

One of the students criticised the method on the ground that it is not proved thereby, that a line cutting a curve at two points at infinity is a tangent to the curve or the limiting position of a tangent.

To say, as is sometimes done, that it works well to consider the two points of intersections at infinity as coincident, is not very satisfactory to the sophomore mind. No proof that the above method gives the limiting position of a tangent line is carried out, so far as I know, in a beginning course in analytic geometry.

The method that merely determines the parameters m and b in $y=mx+b$ by requiring this line to have two of its intersections with the curve $f(x, y)=0$ at infinity, is a direct application of the third definition, and makes no use of the idea of a tangent. It therefore seems to be a natural method to give the third definition when the above method of finding asymptotes is to be used. The fact that it works well to regard an asymptote thus determined as the limiting position of a tangent, by treating the two points of intersection at infinity as coincident, could be brought to the attention of students very naturally without making the idea of the tangent of such primary importance as to include it in the definition. In this connection, it may well be shown by use of the formula,

$$x=\frac{x_1+rx_2}{1+r},$$

for the division of a line segment, that, for fixed x_1 and x_2, the formula gives one and only one value of x that corresponds to a value of r; and that r can take any value except -1. If $r=-1$, the formula is meaningless, but it is convenient to use the symbol ∞ to correspond to $r=-1$, and to look upon the line of which x_1x_2 is a segment as having only one point at infinity to correspond to $r=-1$.

Such a method seems, to the writer, much more natural than that in which the first definition is employed.